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Fluctuations for Kawasaki Dynamics

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In this paper Kawasaki dynamics are considered. Lower bounds are obtained for the variance of the occupation time of a site in any dimension and for temperature above critical temperature. These lower bounds are expressed in terms of the density correlation function and hence relate the fluctuations to some phase transition quantities. At critical temperature, under a reasonable assumption of the static structure function, lower bounds for the variance of the occupation time are obtained. These lower bounds are consistent with the supposed value of the critical exponent. This paper also examines the same problem for Glauber dynamics and shows that the phase transition may not be of importance for the behavior of fluctuations.

KEY WORDS: Kawasaki dynamics; critical dynamics.

1. INTRODUCTION

In this article we are interested in the variance of the occupation time of a site for an interacting particle system known as Kawasaki dynamics. Formally, the dynamics is a Markov process whose state space (or configuration space) is $\Omega = \{0, 1\}^{\mathbb{Z}^d}$. A configuration η describes the occupation of sites in the sense that $\eta(x) = 1$ if there is a particle on site x and $\eta(x) = 0$ otherwise. This interacting particle system $(\eta_t)_{t \geqslant 0}$ consists of particles performing random walks over sites of $\mathbb{Z}^d$ with jump rates depending on the interaction with the other particles and satisfying the exclusion rule: there is at most one particle by site. Consequently, a particle sitting on site x jumps to site y with rate $c(x, y, \eta)$ only if the site y is not occupied by an another particle (otherwise the jump is canceled). For sufficiently high temperature β^{-1}, if the jump rates $c(x, y, \eta)$ satisfy detailed

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balance condition, there exists a unique Gibbs measure $\mu_{\rho,\beta}$ (depending on the density ρ and the inverse temperature β) under which the dynamics is reversible and hence invariant (see Section 2 for a precise statement).

Let us fix the density ρ and the inverse temperature β and consider the gas in thermal equilibrium under the Gibbs measure $\mu_{\rho,\beta}$. We will often omit the index β (or ρ), when the temperature and density will be fixed. The expectation with respect to $\mu_{\rho,\beta}$ is denoted by $\langle\cdot\rangle$. The quantity of interest is the density–density correlation function

$$u_t(x) = \langle \eta_t(x)\eta_0(0)\rangle - \rho^2 \tag{1.1}$$

The Fourier transform of u_t, also known as structure function in the physical literature, is defined by

$$\hat{u}_t(k) = \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{2i\pi k\cdot x} u_t(x), \quad k \in [0,1]^d \tag{1.2}$$

The static compressibility $\chi = \chi(\rho,\beta)$ is given by

$$\chi = \sum_{x\in\mathbb{Z}^d} u_0(x) = \hat{u}_0(0)$$

This quantity is well defined for $\beta < \beta_{\mathrm{c}}$ where β_{c} is the inverse critical temperature defined as $\beta_{\mathrm{c}} = \sup\{\beta;\ \chi(\rho,\beta) < +\infty\}$.

For general Kawasaki dynamics, the knowledge about the density–density correlation function is limited. Nevertheless, we know that time correlations cannot decay exponentially because of the conservation law (cf. ref. 13, p. 176)

$$u_t(0) = \langle \eta_t(0)\eta_0(0)\rangle - \rho^2 \geqslant ct^{-d/2}$$

In this article, we are not directly interested in the density–density correlation function but in the time t variance σ_t^2 of the occupation time of a site. This last quantity is related to the density–density correlation function by the following formula

$$\sigma_t^2 = \mathbb{E}_{\rho,\beta}\left[\int_0^t (\eta_s(0)-\rho)\mathrm{d}s\right]^2 = 2\int_0^t (t-s)u_s(0)\mathrm{d}s \tag{1.3}$$

where $\mathbb{E}_{\rho,\beta}$ stands for the expectation with respect to the law of the process $(\eta_t)_{t\geqslant 0}$ starting from $\mu_{\rho,\beta}$.
Here is our main theorem.

Theorem 1.1. If $\beta < \beta_c$, we have the following lower bounds for the Laplace transform of the time t variance σ_t^2:

$$\liminf_{\lambda \to 0} n(\lambda) \int_0^{+\infty} e^{-\lambda t} \sigma_t^2 dt \geqslant \begin{cases} C_1 \chi(\rho)^{3/2} & \text{for } d=1 \\ C_2 \chi(\rho)^2 & \text{for } d=2 \\ C_d \int_{k\in[0,1]^d} \frac{\hat{u}_0^2(k)}{\sum_{j=1}^d \sin^2(\pi k_j)} dk & \text{for } d \geqslant 3 \end{cases}$$

where C_d is a positive constant independent of ρ, β and the normalization function $n(\lambda)$ satisfies

$$n(\lambda) = \begin{cases} \lambda^{3/2} & \text{for } d=1 \\ \lambda^2(-\log\lambda)^{-1} & \text{for } d=2 \\ \lambda^2 & \text{for } d \geqslant 3 \end{cases}$$

In a Tauberian sense, this theorem means that

$$\begin{cases} \liminf\{t^{-3/2}\sigma_t^2\} > 0 & \text{for } d=1 \\ \liminf\{(t\log t)^{-1}\sigma_t^2\} > 0 & \text{for } d=2 \\ \liminf\{t^{-1}\sigma_t^2\} > 0 & \text{for } d \geqslant 3 \end{cases}$$

Remark that the normalization function is the same as the function given by Kipnis[(4)] for the simple symmetric exclusion process (see Section 4).

More interesting than the normalization function is the dependence on the compressibility (and on the structure function for the dimension $d \geqslant 3$) of the lower bounds. These bounds are valid for any temperature greater than the critical temperature. In dimension $d=1, 2$, they are clearly divergent as $\beta \to \beta_c$. The case of the dimension $d \geqslant 3$ is more intricate. Indeed, it is conjectured (ref. 13, pp. 209–210) that for $\beta = \beta_c$, the static structure function is of the following form

$$\hat{u}_0(k) \sim_{k\to 0} \|k\|^{-2+\eta} \tag{1.4}$$

with the critical exponent $\eta = 0.03$ for $d=3$ and $\eta = 0$ for $d \geqslant 4$. Assuming this fact, we obtain that the lower bounds remain finite as β goes to β_c if and only if $d \geqslant 7$. In fact, at critical temperature, assuming that (1.4) reflects the real behavior of the static structure function, we are able to give lower bounds for the Laplace transform of the time t variance of the occupation time. This is the content of the following theorem

Theorem 1.2. Assume that at inverse critical temperature β_c, the static critical structure function is of the form

$$\hat{u}_0(k) = |k\|^{-2+\eta}\Phi(k)$$

where Φ is a bounded continuous function such that $\Phi(0) > 0$ and η is the static critical exponent given by $\eta = 1/4$ for $d = 2$, $\eta = 0.03$ for $d = 3$ and $\eta = 0$ for $d \geqslant 4$.
Then we have

$$\lim_{\lambda\to 0}\inf n(\lambda)\int_0^{\infty} dt e^{-\lambda t}\sigma_t^2 dt > 0$$

where $n(\lambda)$ is given by

$$n(\lambda) = \begin{cases} \lambda^{38/15} & \text{for } d=2 \\ \lambda^{(11-4\eta)/(4-\eta)} & \text{for } d=3 \\ \lambda^{5/2} & \text{for } d=4 \\ \lambda^{9/4} & \text{for } d=5 \\ \lambda^2|\log(\lambda)|^{-1} & \text{for } d=6 \\ \lambda^2 & \text{for } d\geqslant 7 \end{cases}$$

The paper is organized as follows: in Section 2, we recall some facts about Kawasaki dynamics. In Section 3, we give the proof of Theorem 1.1. The Section 4 is devoted to some remarks concerning the generalized duality which is at the origin of the proof of our results. In the Section 5, we obtain Theorem 1.2. In the last section, we examine the occupation time of a site for Glauber dynamics.

2. KAWASAKI DYNAMICS

In this section, we recall general facts presented in ref. 13. The state space for the lattice gas is $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ where d is the dimension of the lattice. A configuration η describes the occupation of the sites in the sense that $\eta(x) = 1$ if there is a particle on site x and $\eta(x) = 0$ otherwise. The dynamics of the lattice is defined through the jump rates $c(x, y, \eta) = c(y, x, \eta)$: with rate $c(x, y, \eta)$, there is exchange of the occupancies at the pair of sites x and y. The generator of the Markov jump process is given by

$$(\mathcal{L}f)(\eta) = \sum_{x,y\in\mathbb{Z}^d} c(x, y, \eta)\left[f(\eta^{xy}) - f(\eta)\right] \tag{2.1}$$

where f is a local function on Ω and η^{xy} is as usual the configuration obtained from η by exchange of the occupation variables $\eta(x)$ and $\eta(y)$.

We will assume the following conditions for the nonnegative jump rates:

- There exists a finite positive number R such that

$$c(x, y, \eta) = 0 \ \text{ whenever } |x-y| > R \tag{2.2}$$

and such that $c(x, y, \eta)$ depends on η only through $\{\eta(u)/|x-u| \leqslant R, |y-u| \leqslant R\}$.

- Let τ_a be the shift by $a \in \mathbb{Z}^d$ defined by $(\tau_a \eta)(x) = \eta(x-a)$. For all $a, x, y \in \mathbb{Z}^d$ and $\eta \in \Omega$

$$c(x, y, \eta) = c(x+a, y+a, \tau_a \eta) \tag{2.3}$$

By convention, we will take $c(x, y, \eta) = 0$ if $\eta(x) = \eta(y)$ and to avoid degeneracies, we assume that $c(x, y, \eta) > 0$ for $|x - y| = 1, \eta(x) \neq \eta(y)$. Under these conditions, the Markov process $(\eta_t)_{t \geqslant 0}$ with generator defined by Eq. (2.1) is well defined and the set of local functions is a core for $\mathcal{L}$ (cf. ref. 6).

Consider a finite range and translation invariant ferromagnetic potential $(J_A)_{A \subset \mathbb{Z}^d}$ with inverse temperature given by $\beta > 0$. The formal Hamiltonian of the dynamics is

$$H(\eta) = \sum_{A \subset \mathbb{Z}^d} J_A \eta^A$$

where for a finite subset A of $\mathbb{Z}^d$

$$\eta^A = \prod_{z \in A} \eta(z)$$

Of course, the Hamiltonian is not defined in infinite volume but the energy difference between the configurations η^{xy} and η is meaningful:

$$(\Delta_{xy} H)(\eta) = - \sum_{A, x, y \notin A} (J_{A \cup \{x\}} - J_{A \cup \{y\}})(\eta(x) - \eta(y))\eta^A \tag{2.4}$$

The jump rates are supposed to satisfy the following detailed balance condition:

$$c(x, y, \eta) = c(x, y, \eta^{xy}) \exp[-\beta(\Delta_{xy} H)(\eta)] \tag{2.5}$$

To simplify, we will assume that $c(x, y, \eta)$ is a bounded function $\Phi(\beta \Delta_{xy} H(\eta))$ of the energy difference times the inverse temperature β (for example, $\Phi(E) = (1 + \mathrm{e}^{E})^{-1}$).

The lattice gas is considered under a shift invariant Gibbs state μ associated to potential $(J_A)_A$ and temperature β^{-1}. It means that μ is a probability on Ω satisfying the following DLR equations

$$\mu\left(\{\eta(x) = 1 \mid \eta_{\{x\}^c}\}\right) = \left(1 + \exp\left[\beta \sum_{x \in A} J_{A \setminus \{x\}} \eta^{A \setminus \{x\}}\right]\right)^{-1}$$

where $\eta_{\{x\}^c}$ is an arbitrary outside configuration on $\{x\}^c$.

In the sequel we will denote by ρ the conserved density and the Gibbs measure with density ρ and temperature β^{-1} by $\mu_{\rho,\beta}$. Remark that the detailed balance condition gives the reversibility of the process under $\mu_{\rho,\beta}$. The expectation with respect to $\mu_{\rho,\beta}$ will be denoted by $\langle \cdot \rangle_{\rho,\beta}$. The subscripts ρ, β will be often omitted when there will be no confusions.

We will assume the Gibbs measure $\mu_{\rho,\beta}$ satisfies the following exponential mixing for $\beta < \beta_{\mathrm{c}}$: There exist constants $C = C(\beta, \rho), \gamma = \gamma(\beta, \rho) > 0$ such that

$$\left| \langle \eta(x)\eta(0) \rangle - \rho^2 \right| \leqslant C \mathrm{e}^{-\gamma |x|} \tag{2.6}$$

This reasonable assumption has not been proved for any potential $(J_A)_A$. Nevertheless, it has been established for $(J_A)_A$ such that $J_A = 0$ if $|A| \geqslant 3$ (e.g. the Ising model) and in a more general context, for one-component spin systems with distribution belonging to the Griffiths–Simon class and decaying faster than the Gaussians (cf. ref. 1).

Under this assumption, the static structure function $\hat{u}_0$ and the compressibility $\chi(\rho) = \sum_{x \in \mathbb{Z}^d} (\langle \eta(x)\eta(0) \rangle - \rho^2)$ are well defined. Since the potential is assumed ferromagnetic, we have (cf. Theorem 1.21, chapter IV, of ref. (7)):

$$\forall x \in \mathbb{Z}^d, \quad \langle \eta(x)\eta(0) \rangle - \rho^2 \geqslant 0$$

Hence, the compressibility $\chi(\rho)$ is a nonnegative function of the density ρ. Moreover, if $\chi(\rho)=0$ then $\langle\eta(0)^2\rangle-\rho^2=\langle\eta(0)\rangle-\rho^2=\rho-\rho^2=0$ and hence $\rho=0$ or $\rho=1$. Since these cases are irrelevant, we will assume in the sequel that the density ρ belongs to $(0,1)$. Hence, the compressibility $\chi(\rho)$ is positive. The density ρ and inverse temperature β are now fixed.

The idea of the proof of the main theorem is inspired from generalized duality that we explain briefly in Section 4. Nevertheless, the paper is self-contained without this section. Hence, we introduce here some notations of Section 4 in our context. Let $\mathcal{E}$ be the class of all finite subsets of $\mathbb{Z}^d$ and $\mathcal{E}_n$ the subsets of $\mathbb{Z}^d$ with n points. For each $A\in\mathcal{E}$, let Ψ_A be the local function

$$\Psi_A(\eta)=\prod_{x\in A}(\eta(x)-\rho)$$

and by convention $\Psi_\emptyset=1$. It is easy to check that $\{\Psi_A;A\in\mathcal{E}\}$ is a basis of $\mathbb{L}^2(\mu_{\rho,\beta})$. We will denote by $\mathcal{F}_n$ the subspace generated by $\{\Psi_A;A\in\mathcal{E}_n\}$. The functions of $\mathcal{F}_n$ are called functions of degree n.

If f is any element of $\mathbb{L}^2(\mu_{\rho,\beta})$ then we can decompose it in the basis $\{\Psi_A;A\in\mathcal{E}\}$ and we have

$$f=\sum_{n\geqslant 0}\sum_{A\in\mathcal{E}_n}\mathfrak{f}(A)\Psi_A \tag{2.7}$$

Denote by $\langle\cdot,\cdot\rangle$ the inner product on $\mathbb{L}^2(\mu_{\rho,\beta})$. Remark that if f,g are two elements of $\mathbb{L}^2(\mu_{\rho,\beta})$, then we have

$$\langle f,g\rangle=\sum_{A,B\in\mathcal{E}}\mathfrak{f}(A)\mathfrak{g}(B)J(A,B) \tag{2.8}$$

where $J(A,B)=\langle\Psi_A,\Psi_B\rangle$. We note the $\mathbb{L}^2$ norms by

$$\|f\|_0^2=\|\mathfrak{f}\|_0^2=\sum_{A,B\in\mathcal{E}}\mathfrak{f}(A)\mathfrak{f}(B)J(A,B)$$

Let $\mathfrak{G}_n$ be the subspace generated by finite supported functions of degree n. Remark that $\mathfrak{G}_1$ is the set of local functions from $\mathbb{Z}^d$ into $\mathbb{R}$.

In general, this basis is not orthogonal and spaces $\mathcal{F}_n$ are not invariant under the action of $\mathcal{L}$. Nevertheless, it is the case up to a renormalization factor for the symmetric simple exclusion process.

3. OCCUPATION TIME OF A SITE IN A REVERSIBLE LATTICE GAS WITH HARD CORE EXCLUSION

Let us recall that the time t variance of the site 0 is given by

$$\sigma_t^2 = \mathbb{E}_{\rho,\beta}\left[\int_0^t (\eta_s(0)-\rho)\mathrm{d}s\right]^2$$

We follow the method given in ref. 2. First, we express the Laplace transform of the time t variance in terms of the generator. Secondly, we give a variational formula for this expression. Using the basis $\{\Psi_A;\ A\in\mathcal{E}\}$, a lower bound for the variational formula is obtained. We recall that the Laplace transform of the time t variance σ_t^2 is given by (cf. ref. 11):

$$\int_0^{+\infty} \mathrm{e}^{-\lambda t}\sigma_t^2\,\mathrm{d}t = 2\lambda^{-2}\langle(\eta(0)-\rho),(\lambda-\mathcal{L})^{-1}(\eta(0)-\rho)\rangle \tag{3.1}$$

Lemma 3.1. There exist positive constants $C=C(d)$ independent of β,ρ such that

$$\liminf_{\lambda\to 0}\left\{m(\lambda)\langle(\eta(0)-\rho),(\lambda-\mathcal{L})^{-1}(\eta(0)-\rho)\rangle\right\}$$
$$\geqslant \begin{cases} C_1\chi(\rho)^{3/2} & \text{for } d=1\\ C_2\chi(\rho)^2 & \text{for } d=2\\ C_d\displaystyle\int_{[0,1]^d}\frac{\hat{u}_0^2(k)}{\sum_{j=1}^d\sin^2(\pi k_j)}\mathrm{d}k & \text{for } d\geqslant 3\end{cases}$$

where the normalization factor $m(\lambda)$ is defined by

$$m(\lambda)=\begin{cases}\lambda^{1/2} & \text{for } d=1\\ (-\log(\lambda))^{-1} & \text{for } d=2\\ 1 & \text{for } d\geqslant 3\end{cases}$$

Proof. Let us introduce some notations. The Dirichlet form $\mathcal{D}(f)=-\langle f,\mathcal{L}f\rangle$ associated to the process $(\eta_t)_{t\geqslant 0}$ is given by

$$\mathcal{D}(f)=\frac{1}{2}\sum_{x,y\in\mathbb{Z}^d}\langle c(x,y,\eta)\left[f(\eta^{xy})-f(\eta)\right]^2\rangle \tag{3.2}$$

Since the jump rates are uniformly bounded, there exists a positive constant M such that

$$\mathcal{D}(f) \leqslant M \left\langle \sum_{|x-y|=1} \left[f(\eta^{xy}) - f(\eta)\right]^2 \right\rangle \tag{3.3}$$

We have the following variational formula (cf. Lemma 2.1 of ref. 2):

$$\langle f, (\lambda - \mathcal{L})^{-1} f \rangle = \sup_g \{2\langle f, g\rangle - \langle g, (\lambda - \mathcal{L}) g\rangle\} \tag{3.4}$$

$$= \sup_g \left\{2\langle f, g\rangle - \lambda \|g\|_0^2 - \mathcal{D}(g)\right\} \tag{3.5}$$

where the supremum is taken over all local functions g. To obtain a lower bound, we can restrict this supremum over the functions of degree one and use the inequality (3.3). If g is a local function of degree one then $g = \sum_{x\in\mathbb{Z}^d} \mathfrak{g}(x)\Psi_x$. It is easy to see that

$$\mathcal{D}(g) \leqslant M \sum_{|x-y|=1} (\mathfrak{g}(x) - \mathfrak{g}(y))^2 \tag{3.6}$$

We have hence the following estimates for the function $(\eta(0) - \rho) = \Psi_0(\eta)$

$$\begin{aligned}
&\langle(\eta(0) - \rho), (M\lambda - \mathcal{L})^{-1}(\eta(0) - \rho)\rangle \\
&\geqslant \sup_g \left\{ 2\sum_{x\in\mathbb{Z}^d} J(0,x)\mathfrak{g}(x) - M\lambda \sum_{x,y\in\mathbb{Z}^d} \mathfrak{g}(x) J(x,y)\mathfrak{g}(y) - M \sum_{|x-y|=1} (\mathfrak{g}(x) - \mathfrak{g}(y))^2 \right\} \\
&\geqslant \frac{1}{M} \sup_{\mathfrak{g}} \left\{ 2\sum_{x\in\mathbb{Z}^d} u_0(x)\mathfrak{g}(x) - \lambda \sum_{x,y\in\mathbb{Z}^d} \mathfrak{g}(x) u_0(y-x)\mathfrak{g}(y) \right. \\
&\qquad \left. - \sum_{|x-y|=1} (\mathfrak{g}(x) - \mathfrak{g}(y))^2 \right\}
\end{aligned} \tag{3.7}$$

where we recall that the density correlation function u_0 is defined by

$$u_0(x) = J(0,x) = \langle \eta(0)\eta(x)\rangle - \rho^2 \tag{3.8}$$

Of course, the last supremum can be extended to functions $\mathfrak{g}$ such that $\sum_{x\in\mathbb{Z}^d} \mathfrak{g}^2(x) < +\infty$. If $\phi(x)$ is an integrable function over $\mathbb{Z}^d$ (for the counting measure), we define its Fourier transform by

$$\hat{\phi}(k) = \sum_{x\in\mathbb{Z}^d} \phi(x) \exp(2i\pi k\cdot x) \tag{3.9}$$

where $k \in [0,1]^d$.

Thanks to exponential mixing (2.6), the static structure function $\hat{u}_0$ is well defined and smooth. Since $\hat{u}_0$ is a real function, $\hat{u}_0$ is even. Remark that $\hat{u}_0(0) = \chi(\rho)$ is positive because the density belongs to $(0, 1)$. In fact, the static structure function has the following form

$$\hat{u}_0(k) = \chi(\rho) + \theta(k)\gamma(k) \tag{3.10}$$

where $\theta(k) = 4\sum_{j=1}^{d} \sin^2(\pi k_j)$ and $\gamma(k)$ is a smooth function. Indeed, since u_0 is even, we can rewrite the static structure function as

$$\hat{u}_0(k) = \chi(\rho) - 2\sum_{x} \sin^2(\pi x \cdot k) u_0(x) \tag{3.11}$$

with $\sum_x |x|^2 |u_0(x)| < +\infty$. A Taylor's expansion gives (3.10).
By Parseval's relation, we get

$$\begin{aligned}
&\langle (\eta(0) - \rho), (M\lambda - \mathcal{L})^{-1}(\eta(0) - \rho)\rangle \\
&\quad \geqslant \frac{1}{M} \sup_{\phi} \left\{ 2\int_{k\in[0,1]^d} \hat{u}_0(k)\hat{\phi}(k)\, \mathrm{d}k - \lambda \int_{k\in[0,1]^d} \hat{u}_0(k)|\hat{\phi}(k)|^2\, \mathrm{d}k \right. \\
&\qquad \left. - \int_{k\in[0,1]^d} \theta(k)|\hat{\phi}(k)|^2\, \mathrm{d}k \right\}
\end{aligned} \tag{3.12}$$

Choose the function ϕ_λ defined by its Fourier transform as

$$\hat{\phi}_\lambda(k) = \frac{\hat{u}_0(k)}{\lambda \hat{u}_0(k) + \theta(k)} \tag{3.13}$$

For λ sufficiently small, the function $\hat{\phi}_\lambda(k)$ is in $\mathbb{L}^2(\mathbb{T}^d)$. Indeed, we have

$$\lambda \hat{u}_0(k) + \theta(k) = \lambda\chi(\rho) + \theta(k)(1 + \lambda\gamma(k)) \tag{3.14}$$

and the function γ is bounded thus for λ sufficiently small, $\lambda\hat{u}_0(k) + \theta(k) \geqslant \lambda\chi(\rho) > 0$.

Moreover, this function is the Fourier transform of a real function since we have $\hat{\phi}_\lambda^{\,*}(1-k_1, \dots, 1-k_d) = \hat{\phi}_\lambda(k_1, \dots, k_d)$ (here $\hat{\phi}_\lambda^{\,*}$ is the complex conjugate of the function $\hat{\phi}_\lambda$). We have then

$$\langle (\eta(0) - \rho), (M\lambda - \mathcal{L})^{-1}(\eta(0) - \rho)\rangle \geqslant \frac{1}{M} \int_{k\in[0,1]^d} \frac{\hat{u}_0^2(k)\mathrm{d}k}{\lambda \hat{u}_0(k) + \theta(k)} \tag{3.15}$$

Let us call $I(\lambda, \hat{u}_0)$ the last integral:

$$I(\lambda, \hat{u}_0) = \int_{k\in[0,1]^d} \frac{\hat{u}_0^2(k)\mathrm{d}k}{\lambda\hat{u}_0(k)+\theta(k)}$$

It is an "explicit" functional of the static structure function $\hat{u}_0$. Define now C_d by

$$C_d = \begin{cases} \liminf_{\lambda\to 0} \lambda^{1/2} I(\lambda, \hat{u}_0) & \text{for } d=1 \\ -\liminf_{\lambda\to 0} (\log\lambda)^{-1} I(\lambda, \hat{u}_0) & \text{for } d=2 \\ \liminf_{\lambda\to 0} I(\lambda, \hat{u}_0) & \text{for } d\geqslant 3 \end{cases} \tag{3.16}$$

For $d=1,2$, the limit above is explicit and depends only on the compressibility χ. For $d\geqslant 3$, it depends on the all values of the function $\hat{u}_0$. More exactly, we have

$$C_d \sim \begin{cases} \chi^{\frac{3}{2}} & \text{for } d=1 \\ \chi^2 & \text{for } d=2 \\ \int_{[0,1]^d} \frac{\hat{u}_0^2(k)}{\theta(k)}\mathrm{d}k & \text{for } d\geqslant 3 \end{cases} \tag{3.17}$$

Indeed, for $d\geqslant 3$, it is trivial since $1/\theta(k)$ is integrable on $[0,1]^d$ and $\hat{u}_0(k)$ is a bounded function. We just give the proof for $d=1$ since the case $d=2$ can be obtained in the same way. Recall Eq. (3.10) and use that $\hat{u}_0$ is even to write

$$\begin{aligned} \lambda^{\frac{1}{2}} I(\lambda, \hat{u}_0) &= 2\int_0^{\frac{1}{2}} \frac{(\chi+\gamma(k)\theta(k))^2}{\lambda\hat{u}_0(k)+\theta(k)}\mathrm{d}k \\ &= 2\chi^2\lambda^{\frac{1}{2}} \int_0^{\frac{1}{2}} \frac{\mathrm{d}k}{\lambda\hat{u}_0(k)+\theta(k)} + 4\chi\lambda^{1/2}\int_0^{\frac{1}{2}} \frac{\gamma(k)\theta(k)}{\lambda\hat{u}_0(k)+\theta(k)}\mathrm{d}k \\ &\quad +2\lambda^{1/2}\int_0^{\frac{1}{2}} \frac{\gamma(k)^2\theta^2(k)}{\lambda\hat{u}_0(k)+\theta(k)}\mathrm{d}k \end{aligned}$$

Since $\gamma, \hat{u}_0$ are smooth functions, the two last terms of the last right hand side go to zero with λ. Just the first term remains

$$J(\lambda, \hat{u}_0) = \lambda^{\frac{1}{2}}\chi^2 \int_0^{\frac{1}{2}} \frac{\mathrm{d}k}{\lambda\hat{u}_0(k)+\theta(k)}$$

Let $\varepsilon > 0$. There exists some $\alpha < 1/2$ (depending on $\hat{u}_0$) such that for $0 \leqslant k \leqslant \alpha$, $\hat{u}_0(k) \leqslant (1+\varepsilon)\chi$. Since $\theta(k) \geqslant \theta(\alpha) > 0$ as soon as $k \geqslant \alpha$, we have

$$\liminf_{\lambda \to 0} J(\lambda, \hat{u}_0) = \liminf_{\lambda \to 0} \lambda^{\frac{1}{2}} \chi^2 \int_0^{\alpha} \frac{\mathrm{d}k}{\lambda \hat{u}_0(k) + \theta(k)}$$

Now,

$$\liminf_{\lambda \to 0} \lambda^{\frac{1}{2}} \chi^2 \int_0^{\alpha} \frac{\mathrm{d}k}{\lambda \hat{u}_0(k) + \theta(k)} \geqslant \liminf_{\lambda \to 0} \lambda^{\frac{1}{2}} \chi^2 \int_0^{\alpha} \frac{\mathrm{d}k}{\lambda(1+\varepsilon)\chi + \theta(k)} \tag{3.18}$$

$$\geqslant \liminf_{\lambda \to 0} \lambda^{\frac{1}{2}} \chi^2 \int_0^{\alpha} \frac{\mathrm{d}k}{\lambda(1+\varepsilon)\chi + \pi^2 k^2} \tag{3.19}$$

$$= (1+\varepsilon)^{\frac{-1}{2}} \chi^{\frac{3}{2}} \int_0^{\infty} \frac{\mathrm{d}t}{1+\pi^2 t^2} \tag{3.20}$$

Taking the limit as ε goes to zero, we get that there exists a constant C_1 independent of λ, β, ρ such that

$$\liminf_{\lambda \to 0} \lambda^{\frac{1}{2}} \langle (\eta(0) - \rho), (\lambda - \mathcal{L})^{-1} (\eta(0) - \rho) \rangle \geqslant C_1 \chi^{\frac{3}{2}}. \quad \blacksquare$$

4. GENERALIZED DUALITY

The main ideas in the proofs of the theorems were inspired by generalized duality methods, which were introduced by Landim and Yau in ref. 5 and developed in several papers. The recent efforts have concentrated essentially on simple exclusion process, the simplest example of Kawasaki dynamics.

One of the reasons for which generalized duality methods have been so powerful in the understanding of the simple exclusion process is that the equilibrium states are given by Bernoulli product measures. It is not clear these methods can be adapted with such great success for general Kawasaki dynamics. In the present article, interesting facts for these dynamics are obtained using only "basic" generalized duality methods. A second step would be to further develop these methods in order to improve the results given here. For example, finding upper bounds for the occupation time of a site would be of high interest.

Following the suggestion of an anonymous referee, we give here a short presentation of these methods which appear often in technically too involved papers. In this section, the focus is on the occupation time of a site for the *symmetric* simple exclusion process. The problem is sufficiently simple to demonstrate in a few lines an important part of the results obtained by Kipnis in ref. 4. The section also briefly mentions how the methods have to be adapted for the *asymmetric* case.

The simple exclusion process is a Kawasaki dynamics considered at infinite temperature, where the only interaction between particles is due to the exclusion rule (at most one particle by site). More explicitly, it is a Kawasaki dynamics $(\eta_t)_{t\geqslant 0}$ of the form introduced in Section 2 with jump rates $c(x, y, \eta)$ given by

$$c(x, y, \eta) = p(y - x)\eta(x)(1 - \eta(y))$$

where p is an irreducible transition probability on $\mathbb{Z}^d$ with finite range. Let μ_ρ be the Bernoulli product measure with parameter $\rho \in [0, 1]$ on Ω. If p is symmetric ($p(x) = p(-x)$ for any $x \in \mathbb{Z}^d$), the process $(\eta_t)_{t\geqslant 0}$ is reversible under the Bernoulli product measure μ_ρ for any $\rho \in [0, 1]$. Here, the parameter ρ is in fact the conserved density of particles. If p is asymmetric (i.e. nonsymmetric) then for any ρ the Bernoulli product measure μ_ρ remains invariant for the process but it is no longer reversible.

As in Section 2, we introduce the class $\mathcal{E}$ of finite subsets of $\mathbb{Z}^d$ and the class $\mathcal{E}_n$ of subsets of $\mathbb{Z}^d$ with n points. We fix $\rho \in (0, 1)$. For each $A \in \mathcal{E}$, let Ψ_A be the local function

$$\Psi_A(\eta) = \prod_{x\in A} \frac{(\eta(x) - \rho)}{\sqrt{\rho(1-\rho)}}$$

and by convention $\Psi_\emptyset = 1$. It is easy to check that $\{\Psi_A; A \in \mathcal{E}\}$ is an *orthonormal* basis of $\mathbb{L}^2(\mu_\rho)$. This property is particular to Bernoulli product measures and is false if μ_ρ is replaced by a Gibbs measure associated to a general potential $(J_A)_A$ as in Section 2. Indeed, the random variables $(\eta(x) - \rho)$ are in general correlated in this case.

We will denote by $\mathcal{F}_n$ the subspace generated by $\{\Psi_A; A \in \mathcal{E}_n\}$. The functions of $\mathcal{F}_n$ are called functions of degree n. If f is any element of $\mathbb{L}^2(\mu_\rho)$ then we can decompose it in the basis $\{\Psi_A; A \in \mathcal{E}\}$ and we have

$$f = \sum_{n\geqslant 0} \sum_{A\in\mathcal{E}_n} \mathfrak{f}(A)\Psi_A \tag{4.1}$$

Denote by $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{L}^2(\mu_\rho)$. If f, g are two elements of $\mathbb{L}^2(\mu_\rho)$, then

$$\langle f, g \rangle = \sum_{A \in \mathcal{E}} \mathfrak{f}(A)\mathfrak{g}(A) \tag{4.2}$$

We adopt the following notations for the $\mathbb{L}^2$ norms

$$\| f \|_0^2 = \| \mathfrak{f} \|_0^2 = \sum_{A \in \mathcal{E}} \mathfrak{f}(A)^2$$

Let $\mathfrak{G}_n$ be the subspace generated by finite supported functions of degree n. Remark that $\mathfrak{G}_1$ is just the set of local functions from $\mathbb{Z}^d$ into $\mathbb{R}$. We have:

$$\mathbb{L}^2(\mathcal{E}) = \bigoplus_{n \geqslant 1} \mathfrak{G}_n.$$

The generator $\mathcal{L}$ of the simple exclusion process acts on the local functions f of $\mathbb{L}^2(\mu_\rho)$ as:

$$(\mathcal{L} f)(\eta) = \sum_{x, y \in \mathbb{Z}^d} p(y - x)\eta(x)(1 - \eta(y))(f(\eta^{x,y}) - f(\eta))$$

and gives hence an operator $\mathfrak{L}$ on $\mathbb{L}^2(\mathcal{E})$. More exactly, if f is a local function with the following decomposition

$$f = \sum_{A \in \mathcal{E}} \mathfrak{f}(A)\Psi_A$$

then

$$\mathcal{L} f = \sum_{A \in \mathcal{E}} (\mathfrak{L}\mathfrak{f})(A)\Psi_A$$

Before giving the explicit form of the operator $\mathfrak{L}$, we need to introduce notations. For a subset A of $\mathbb{Z}^d$ and x, y in $\mathbb{Z}^d$ denote by $A_{x,y}$ the set defined by $A_{x,y} = A \backslash \{x\} \cup \{y\}$ if $x \in A$ and $y \notin A$, $A_{x,y} = A \backslash \{y\} \cup \{x\}$ if $y \in A$ and $x \notin A$ and $A_{x,y} = A$ otherwise. Let $s(\cdot)$ and $a(\cdot)$ be the symmetric and antisymmetric part of the transition probability $p(\cdot)$. In the basis $\{\Psi_A; A \in \mathcal{E}\}$, we have the following decomposition of the operator $\mathfrak{L}$:

$$\mathfrak{L} = \mathfrak{T} + \mathfrak{A}$$

where $\mathfrak{A} = (1 - 2\rho)\mathfrak{L}^2 + 2\sqrt{\rho(1 - \rho)}(\mathfrak{L}^+ - \mathfrak{L}^-)$, with

$$
\begin{aligned}
(\mathfrak{T}\mathfrak{f})(A) &= (1/2) \sum_{x,y\in\mathbb{Z}^d} s(y-x)\left[\mathfrak{f}(A_{x,y}) - \mathfrak{f}(A)\right] \\
(\mathfrak{L}^2\mathfrak{f})(A) &= \sum_{x\in A, y\notin A} a(y-x)\left[\mathfrak{f}(A_{x,y}) - \mathfrak{f}(A)\right] \\
(\mathfrak{L}^-\mathfrak{f})(A) &= \sum_{x\notin A, y\notin A} a(y-x)\mathfrak{f}(A\cup\{x\}) \\
(\mathfrak{L}^+\mathfrak{f})(A) &= \sum_{x\in A, y\in A} a(y-x)\mathfrak{f}(A-\{y\})
\end{aligned}
$$

For example, consider the symmetric simple exclusion process for which $s = p$ and $a = 0$. We have $\mathfrak{L} = \mathfrak{T}$ and $\mathfrak{A} = 0$. The expression of $\mathfrak{T}$ shows $\mathfrak{L} = \mathfrak{T}$ lets $\mathfrak{S}_n$ invariant. Moreover, $\mathfrak{L}$ restricted to $\mathfrak{S}_n$ is the infinitesimal generator of n particles in symmetric simple exclusion. This property is known as "duality property" and express that the symmetric simple exclusion process has a dual process which is the symmetric simple exclusion process. This property was already observed by Spitzer but formulated in a different way (cf. ref. 12). The formulation of Spitzer was the following. Consider n particles moving according to symmetric simple exclusion. The location of the particles at time t define a random subset $A_t \in \mathcal{E}_n$. The generator of the Markov process $(A_t)_{t\geqslant 0}$ is $\mathfrak{T}|_{\mathfrak{S}_n}$. Let $\mathbb{P}^A$ denote the law of $(A_t)_{t\geqslant 0}$ starting from $A \in \mathcal{E}_n$ and $\mathbb{P}_\mu$ the law of the symmetric simple exclusion $(\eta_t)_{t\geqslant 0}$ with initial measure μ. Then

$$
\mathbb{P}_\mu\left(\prod_{x\in A}\eta_t(x) = 1\right) = \mathbb{P}^A\left(\mu\left(\prod_{x\in A_t}\eta(x) = 1\right)\right) \tag{4.3}
$$

The formulation here is more algebraic and is applicable to the nonsymmetric case. Indeed, if p is asymmetric, the duality relation Eq. (4.3) is false. The restriction of operator $\mathfrak{L}$ to $\mathfrak{S}_n$ is not the generator of a Markov process. In fact, $\mathfrak{L}$ does not let $\mathfrak{S}_n$ invariant. Nevertheless, the algebraic expression above can be used to make estimates even if the original process has not really got a dual process. The previous Markov generator $\mathfrak{T}$ is just replaced by a complex operator $\mathfrak{T} + \mathfrak{A}$. It explains the term of "generalized duality".

We look now at our initial problem with the symmetric simple exclusion process. Let us recall that the time t variance of the site 0 is given by

$$
\sigma_t^2 = \mathbb{E}_\rho\left[\int_0^t (\eta_s(0) - \rho)\,\mathrm{d}s\right]^2
$$

where $\mathbb{E}_\rho$ denotes the expectation with respect to the law of the symmetric simple exclusion process $(\eta_t)_{t\geqslant 0}$ starting from μ_ρ.

We show here how the generalized duality allows recovery of certain results obtained by Kipnis in ref. (4). As in Section 3, we express the Laplace transform of the time t variance in terms of the generator. Secondly, we give a variational formula for this expression. Using generalized duality, we compute explicitly the value given by the variational formula. We recall that the Laplace transform of the time t variance is given by:

$$\int_0^{+\infty} e^{-\lambda t}\sigma_t^2 dt = 2\lambda^{-2}\langle(\eta(0)-\rho), (\lambda-\mathcal{L})^{-1}(\eta(0)-\rho)\rangle$$
$$= 2\rho(1-\rho)\lambda^{-2}\langle\Psi_{\{0\}}, (\lambda-\mathcal{L})^{-1}\Psi_{\{0\}}\rangle$$

Thanks to the variational formula (3.4), we have

$$\langle\Psi_{\{0\}}, (\lambda-\mathcal{L})^{-1}\Psi_{\{0\}}\rangle = \sup_g \left\{2\langle\Psi_{\{0\}}, g\rangle - \langle g, (\lambda-\mathcal{L})g\rangle\right\}$$

where the supremum is taken over all local functions g. Using generalized duality, this variational formula can be rewritten as

$$\langle\Psi_{\{0\}}, (\lambda-\mathcal{L})^{-1}\Psi_{\{0\}}\rangle = \sup_{\mathfrak{g}} \{2\mathfrak{g}(0) - \langle\mathfrak{g}, (\lambda-\mathfrak{L})\mathfrak{g}\rangle\}$$

Now, the supremum is carried over finite supported functions from $\mathcal{E}$ to $\mathbb{R}$. Recall now the transition probability p is symmetric. Hence, $\mathfrak{L}\mathfrak{g}$ is given by

$$(\mathfrak{L}\mathfrak{g})(A) = (1/2)\sum_{x,y\in\mathbb{Z}^d} p(y-x)\left[\mathfrak{g}(A_{x,y}) - \mathfrak{g}(A)\right]$$

In particular, if $\mathfrak{g}$ has the following decomposition

$$\mathfrak{g} = \sum_{n\in\mathbb{N}} \mathfrak{g}_n$$

where for each $n\in\mathbb{N}$, $\mathfrak{g}_n$ belongs to $\mathfrak{S}_n$, we have

$$\langle\mathfrak{g}, (\lambda-\mathfrak{L})\mathfrak{g}\rangle = \sum_{n\in\mathbb{N}}\langle\mathfrak{g}_n, (\lambda-\mathfrak{L})\mathfrak{g}_n\rangle$$

since $\mathfrak{L}$ let $\mathfrak{S}_n$ invariant and $\mathfrak{S}_n$ is orthogonal to $\mathfrak{S}_m$ for $m \neq n$. Hence, we can restrict the supremum over functions belonging to $\mathfrak{S}_1$ and we obtain

$$\langle \Psi_{\{0\}}, (\lambda - \mathcal{L})^{-1} \Psi_{\{0\}} \rangle = \sup_{\mathfrak{g} \in \mathfrak{S}_1} \{2\mathfrak{g}(0) - \langle \mathfrak{g}, (\lambda - \mathfrak{L})\mathfrak{g} \rangle\}$$

A little computation shows that if $\mathfrak{g} : \mathbb{Z}^d \to \mathbb{R}$ is a finite supported function then

$$\langle \mathfrak{g}, (\lambda - \mathfrak{L})\mathfrak{g} \rangle = \lambda \sum_{x \in \mathbb{Z}^d} \mathfrak{g}^2(x) + \frac{1}{4} \sum_{x, y \in \mathbb{Z}^d} p(y - x) [\mathfrak{g}(x) - \mathfrak{g}(y)]^2$$

The Laplace transform of the time t variance of site 0 is then given by

$$\int_0^{+\infty} e^{-\lambda t} \sigma_t^2 dt$$
$$= 2\rho(1-\rho)\lambda^{-2} \sup_{\mathfrak{g} \in \mathfrak{S}_1} \left\{ \mathfrak{g}(0) - \lambda \sum_{x \in \mathbb{Z}^d} \mathfrak{g}^2(x) - \frac{1}{4} \sum_{x, y \in \mathbb{Z}^d} p(y-x)[\mathfrak{g}(x) - \mathfrak{g}(y)]^2 \right\}$$

This last expression is similar to the expression found in Section 3. Using Fourier calculus, we compute it explicitly (see Section 3 for more details) and we get

$$\frac{2\rho(1-\rho)}{\lambda^2} \int_0^1 \frac{dk}{\lambda + \theta(k)}$$

where $\theta(k) = \sum_{z \in \mathbb{Z}^d} \sin^2(\pi k \cdot z) p(z)$.

There is an important difference with Section 3: we have here an equality between the Laplace transform of the time t variance and the last variational formula. Having the exact expression of the Laplace transform, it is easy to obtain behavior of σ_t as t goes to infinity thanks to classical Tauberian theorems. To simplify notations, assume that $p(z) = (2d)^{-1}$ if $|z| = 1$. We have

$$\begin{cases} \lim_{t \to \infty} t^{-3/2} \sigma_t^2 = \rho(1-\rho) \dfrac{8}{3\sqrt{2\pi}} & \text{for } d = 1 \\ \lim_{t \to \infty} (t \log t)^{-1} \sigma_t^2 = \rho(1-\rho) \dfrac{2}{\pi} & \text{for } d = 2 \\ \lim_{t \to \infty} t^{-1} \sigma_t^2 = 2\rho(1-\rho) \int_0^\infty q_s(0,0)\, ds & \text{for } d \geqslant 3 \end{cases}$$

where $q_s(x, y)$ is the transition probability at time s of the standard d-dimensional symmetric random walk. These last limits were soon obtained by Kipnis.[4]

Consider now the case where the simple exclusion process is not symmetric. We have still a variational formula to express the Laplace transform of the time t variance σ_t^2 but a term due to the antisymmetric part of the generator appears:

$$\int_0^\infty \mathrm{e}^{-\lambda t}\sigma_t^2\,\mathrm{dt} \\ = 2\rho(1-\rho)\lambda^{-2}\sup_{\mathfrak{g}}\left\{\mathfrak{g}(0) - \langle \mathfrak{g}, (\lambda - \mathfrak{T})\mathfrak{g}\rangle - \langle \mathfrak{A}\mathfrak{g}, (\lambda - \mathfrak{T})^{-1}\mathfrak{A}\mathfrak{g}\rangle\right\}$$

In order to obtain a lower bound, we restrict the supremum over degree one functions. Even if $\mathfrak{g} \in \mathfrak{S}_1$, we have $\mathfrak{A}\mathfrak{g} \in \mathfrak{S}_1 \oplus \mathfrak{S}_2$. The presence of degree 2 functions add difficulties to estimate the supremum. In particular, an approximation of particles in exclusion by "free particles" is needed. We refer the interested reader to ref. 2.

5. OCCUPATION TIME AT CRITICAL TEMPERATURE

Theorem 1.1 gives behavior of the occupation time for $\beta < \beta_c$. In this section, we obtain lower bounds for the time t variance of the occupation time σ_t^2 at critical temperature under a reasonable assumption on the static structure function. We will see that our results are consistent with some conjectures formulated in ref. 13, pp. 209–210. Let us denote by $u_t^c(x)$ (resp. $\hat{u}_t^c(k)$) the density–density correlation function (resp. the structure function) at the critical temperature. The critical static structure function, obtained for $t = 0$, is supposed to be of the form

$$\hat{u}_0^c(k) = \|k\|^{-2+\eta}\Phi(k)$$

where Φ is a continuous bounded function with $\Phi(0) > 0$ and η is the critical exponent. In dimension 2, the scaling form above is known with $\eta = 1/4$ from the Onsager solution. For dimension 3, approximate methods give $\eta = 0.03$ and for dimension $d \geqslant 4$, we have $\eta = 0$.

The method presented in Section 3 can be applied without major modification. We have

$$\langle (\eta(0) - \rho), (M\lambda - \mathcal{L})^{-1}(\eta(0) - \rho)\rangle \\ \geqslant \frac{1}{M}\sup_{\phi}\left\{2\int_{k\in[0,1]^d}\hat{u}_0^c(k)\hat{\phi}(k)\mathrm{d}k - \lambda\int_{k\in[0,1]^d}\hat{u}_0^c(k)|\hat{\phi}(k)|^2\mathrm{d}k\right.$$

$$-\int_{k\in[0,1]^d} \theta(k)|\hat{\phi}(k)|^2 \mathrm{d}k\Bigg\} \tag{5.1}$$

Choose the function ϕ_λ defined by its Fourier transform

$$\hat{\phi}_\lambda(k) = \frac{\hat{u}_0^c(k)}{\lambda\hat{u}_0^c(k) + \theta(k)} = \frac{1}{\lambda + \dfrac{\theta(k)}{\hat{u}_0^c(k)}} \tag{5.2}$$

The function $\hat{\phi}_\lambda(k)$ is in $\mathbb{L}^2(\mathbb{T}^d)$ for $\lambda > 0$. Indeed, the only problem is in 0 and we have

$$\hat{\phi}_\lambda(k) \sim_0 \frac{1}{\lambda + 4d\pi^2\Phi(0)^{-1}\|k\|^{4-\eta}} \tag{5.3}$$

Moreover, this function is well the Fourier transform of a real function since we have $\hat{\phi}_\lambda^*(1-k_1, \ldots, 1-k_d) = \hat{\phi}_\lambda(k_1, \ldots, k_d)$ (here $\hat{\phi}_\lambda^*$ is the complex conjugate of the function $\hat{\phi}_\lambda$). We have by similar computations to Section 3

$$\langle(\eta(0)-\rho), (M\lambda - \mathcal{L})^{-1}(\eta(0)-\rho)\rangle \geqslant \frac{1}{M}\int_{k\in[0,1]^d} \frac{(\hat{u}_0^c)^2(k)\,\mathrm{d}k}{\lambda\hat{u}_0^c(k) + \theta(k)} \tag{5.4}$$

After standard analysis, under the hypothesis concerning the critical static structure function, we rigorously obtain some lower bounds for the Laplace transform of the time t variance σ_t^2. Hence, we have proved Theorem 1.2.

The bounds obtained indicate that the limiting variance defined as the limit of $t^{-1}\sigma_t^2$ as t increases to infinity is finite if and only if $d \geqslant 7$.

A more general hypothesis consists to assume that the critical dynamic structure function is of the form

$$\hat{u}_t^c(k) = \|k\|^{-2+\eta}\Psi(\|k\|^z t) \tag{5.5}$$

Here, z is the dynamical critical exponent and Ψ is some function vanishing at infinity such that $\Psi(0) = 1$. The Gaussian approximation (cf. ref. 13, pp. 209–210) suggests that

$$\hat{u}_t^c(k) = \hat{u}_0^c(k)\exp\left[-\sigma_c k^2|t|/\hat{u}_0^c(k)\right]$$

where σ_c is the conductivity of the lattice gas. Hence the critical exponent z should be

$$z = 4 - \eta$$

But, in an another way, it is easy to show that the limiting variance σ_∞^2 is equal to (cf. ref. 10):

$$\sigma_\infty^2 = \lim_{t\to\infty} t^{-1}\sigma_t^2 = 2\int_0^\infty u_s^c(0)\mathrm{d}s \tag{5.6}$$

Assuming that we can use the inversion Fourier transform formula, we get

$$\sigma_\infty^2 = \int_0^\infty \mathrm{d}t \int_{[0,1]^d} dk\hat{u}_t^c(k) \tag{5.7}$$

If the critical exponent is $z = 4 - \eta$ then Eq. (5.5) gives

$$\begin{aligned}\int_0^\infty \mathrm{d}t \int_{[0,1]^d} \mathrm{d}k\hat{u}_t^c(k) &= \int_0^\infty \mathrm{d}t \int_{[0,1]^d} \mathrm{d}k \|k\|^{-2+\eta}\Psi(\|k\|^{4+\eta}t)\\ &= C_d\left(\int_0^\infty \mathrm{d}s\Psi(s)\right)\int_0^1 r^{-7+2\eta+d}\mathrm{d}r\end{aligned} \tag{5.8}$$

where C_d is the volume of the unit sphere of $\mathbb{R}^d$. Using the assumed value of η, it appears that this last quantity if finite if and only if $d \geqslant 7$. This fact is consistent with our previous result.

6. SOME REMARKS ON THE OCCUPATION TIME OF A SITE FOR GLAUBER DYNAMICS

In this section, we consider a Glauber dynamics $(\eta_t)_{t\geqslant 0}$ with inverse temperature $\beta > 0$. The state space is now $\Omega = \{-1, 1\}^{\mathbb{Z}^d}$ and the generator of this Markov process $(\eta_t)_{t\geqslant 0}$ is given by

$$(\mathcal{L}_\beta f)(\eta) = \sum_{x\in\mathbb{Z}^d} c_\beta(x,\eta)\left[f(\eta^x) - f(\eta)\right] \tag{6.1}$$

where f is a local function on Ω and η^x is as usual the configuration obtained from η by replace $\eta(x)$ by $-\eta(x)$. The formal Hamiltonian of this model is

$$H_\beta(\eta) = \beta \sum_{|x-y|=1} \eta(x)\eta(y)$$

We assume that the flip rates $\{c_\beta(x,\eta)\,;\, x\in\mathbb{Z}^d\}$ are translation invariant:

$$c_\beta(x,\eta)=c_\beta(0,\tau_x\eta),$$

and upper bounded by some constants independent of β:

$$\sup_{x,\eta}\{c_\beta(x,\eta)\}\leqslant M,$$

and satisfies the detailed balance condition

$$\frac{c_\beta(x,\eta)}{c_\beta(x,\eta^x)}=\exp(-\Delta_x H_\beta(\eta)) \tag{6.2}$$

where the difference energy $\Delta_x H_\beta(\eta) = H_\beta(\eta^x) - H_\beta(\eta) = -2\beta\eta(x)\sum_{|y-x|=1}\eta(y)$ is well defined.

For $d\geqslant 2$ (resp. $d=1$), if β is less than the inverse critical temperature β_c (resp. for all β), there exists a unique Gibbsian measure μ_β with good ergodic properties for which the dynamics given by $\mathcal{L}_\beta$ is reversible. If β is fixed, μ_β is denoted by $\langle\cdot\rangle$ and the corresponding scalar product on $\mathbb{L}^2(\mu_\beta)$ by $\langle\cdot,\cdot\rangle$. As usual, $\mathbb{E}_\beta$ stands for expectation with respect to the law of $(\eta_t)_{t\geqslant 0}$ starting from μ_β.

Minlos[8] obtained for the Ising model (in any dimension and for high temperature) the asymptotics of the correlation functions. the method used is the decomposition of the space where lives the generator $\mathcal{L}_\beta$ into several $\mathcal{L}_\beta$-invariant subspaces. He derives (Theorem 5.1 of ref. 8) the following asymptotics for the correlations between sites x and y:

$$\mathbb{E}_\beta(\eta_0(x)\eta_t(y))\sim_{t\to+\infty}\frac{C(x-y)}{t^{\frac{d}{2}}}\exp(-ct) \tag{6.3}$$

where $C(x)$ is function of the site x and c a positive constant. Nevertheless, this asymptotics are only valid for $\beta<\beta_0$ where β_0 is not explicit. Hence, theses results are useless for β near of the inverse critical temperature.

As for the case of Kawasaki dynamics, let us consider the time t variance σ_t^2 of the occupation time of the site 0:

$$\sigma_t^2=\mathbb{E}_\beta\left[\left(\int_0^t \eta_s(0)\mathrm{d}s\right)^2\right] \tag{6.4}$$

Following the method given in Section 3, lower bounds for the occupation time of a site in Glauber dynamics are given up to the inverse critical temperature β_c. In particular, it means that a diffusive or super diffusive behavior of the occupation time is attempted at critical temperature.

Theorem 6.1. Fix the dimension $d \geqslant 2$. There exists some constant $A > 0$ (independent of β) such that for $\beta < \beta_c$,

$$\liminf_{\lambda \to 0} \lambda^2 \int_0^\infty e^{-\lambda t} \sigma_t^2 dt \geqslant A$$

Proof. Let us denote by $\mathcal{D}_\beta$ the Dirichlet form associated to the generator $\mathcal{L}_\beta$. We have

$$\langle \eta_0, (\lambda - \mathcal{L}_\beta)^{-1} \eta(0) \rangle = \sup_g \{ 2\langle \eta_0, g \rangle - \lambda \langle g^2 \rangle - \mathcal{D}_\beta(g) \}$$

where the supremum is taken over local functions. To obtain a lower bound, we restrict the supremum over the functions of the form $g(\eta) = \sum_x \mathfrak{g}(x)\eta(x)$. Since for a such function,

$$\mathcal{D}_\beta(g) \leqslant 4M \sum_x \mathfrak{g}^2(x)$$

we obtain immediately the theorem. ∎

For the dimension 1, the situation is simpler and we obtain also an upper bound. Indeed, for any inverse of the temperature $\beta > 0$, let $\gamma = \tanh(\beta)$ and introduce the function Ψ_A given by:

$$\Psi_A(\eta) = \prod_{x \in A} \frac{\eta_x - \gamma \eta_{x-1}}{(1-\gamma^2)^{1/2}}$$

where A is a finite subset of $\mathbb{Z}^d$.

$\{\psi_A; A \subset\subset \mathbb{Z}\}$ is an hibertian basis such that the subspaces $\mathcal{H}_n = \{\sum_{|A|=n} \mathfrak{A}\Psi_A\, ;\, \sum_{|A|=n} \mathfrak{f}^2(A) < +\infty\}$ are $\mathcal{L}_\beta$-invariant (cf. ref. 9).

Let us first consider the case where the flip rates $\{c_\beta(x, \cdot)\, ;\, x \in \mathbb{Z}^d\}$ are of the form:

$$c_\beta(x, \eta) = (1 + \exp(-(\Delta_x H_\beta)(\eta)))^{-1}$$

so that the Dirichlet form associated to the generator can be rewritten as:

$$\mathcal{D}_\beta(f) = \langle -\mathcal{L}_\beta f, f\rangle = \sum_{x\in\mathbb{Z}^d} \left\langle \left[f(\eta^x) - f(\eta) \right]^2 \right\rangle$$

In this case, the Laplace transform of the time t variance is explicitly computable and we obtain the following theorem:

Theorem 6.2. If the flip rates are of the form $c_\beta(x,\eta) = (1 + \exp(-(\Delta_x H_\beta)(\eta)))^{-1}$ then

$$\int_0^\infty \sigma_t^2 \mathrm{e}^{-\lambda t}\mathrm{d}t = \frac{2(1-\gamma^2)}{\lambda^2} \int_0^1 \left(\lambda + \frac{4}{1-\gamma^2} |1-\gamma \exp(-2i\pi s)|^2 \right)^{-1} \times |1-\gamma\exp(-2i\pi s)|^{-2}\,\mathrm{d}s$$

and in particular,

$$\lim_{\lambda\to 0} \lambda^2 \int_0^\infty \sigma_t^2 \mathrm{e}^{-\lambda t}\,\mathrm{d}t = 12 \left(\frac{1+\gamma^2}{1-\gamma^2} \right)$$

Proof. We have still the following variational formula for $L(\lambda) = \langle \eta_0, (\lambda - \mathcal{L}_\beta)^{-1}\eta_0\rangle$:

$$L(\lambda) = \sup_f \left\{ 2\langle f, \eta(0)\rangle - \lambda\langle f^2\rangle - \mathcal{D}_\beta(f) \right\}$$

For each $a\in\mathbb{Z}$, $\eta(a)$ is a function belonging to $\mathcal{H}_1$ whose decomposition on the basis is:

$$\eta(a) = (1-\gamma^2)^{1/2} \sum_{k\geqslant 0} \gamma^k \Psi_{a-k} \tag{6.5}$$

In particular, $\eta(0)$ is a degree one function. Hence, the supremum in the preceding variational formula can be carried over the set of local functions belonging to $\mathcal{H}_1$. Using (6.5), if $f = \sum_{k\in\mathbb{Z}} \phi(k)\Psi_k$ is a degree one function, we have:

$$\mathcal{D}_\beta(f) = \frac{4}{1-\gamma^2} \sum_{k\in\mathbb{Z}} [\gamma\phi(k+1) - \phi(k)]^2 \tag{6.6}$$

and

$$2\langle f, \eta_0 \rangle = 2\sqrt{1-\gamma^2}\sum_{k=0}^{+\infty} \gamma^k \phi(-k) \tag{6.7}$$

Hence we have to maximize the following expression over ϕ

$$2\sqrt{1-\gamma^2}\sum_{k=0}^{+\infty} \gamma^k \phi(-k) - \lambda \sum_{k\in\mathbb{Z}} \phi(k)^2 - 4(1-\gamma^2)^{-1} \sum_{k\in\mathbb{Z}} [\gamma\phi(k+1) - \phi(k)]^2 \tag{6.8}$$

As in the preceding section, we use Fourier calculus to express (6.8) in the following form

$$\begin{aligned} &2\sqrt{1-\gamma^2}\int_0^1 \frac{\hat{\phi}(s)}{1-\gamma\exp(-2i\pi s)}\mathrm{d}s \\ &\quad - \int_0^1 \left(\lambda + 4(1-\gamma^2)^{-1}\left|1-\gamma\exp(-2i\pi s)\right|^2\right)\left|\hat{\phi}(s)\right|^2 \mathrm{d}s \end{aligned} \tag{6.9}$$

Let $\hat{\psi}(s) = (1-\gamma\exp(-2i\pi s))^{-1}\hat{\phi}(s)$. We have therefore to maximize over functions ψ the integral of a quadratic expression of $\hat{\psi}$:

$$\int_0^1 \left\{ 2\sqrt{1-\gamma^2}\hat{\psi}(s) - \left(\lambda + 4(1-\gamma^2)^{-1}\left|1-\gamma\exp(-2i\pi s)\right|^2\right) \left|1-\gamma\exp(-2i\pi s)\right|^2 \left|\hat{\psi}(s)\right|^2 \right\} \mathrm{d}s$$

The supremum of this functional over the functions ψ is given by

$$\begin{aligned} &(1-\gamma^2)\int_0^1 \left(\lambda + \frac{4}{1-\gamma^2}\left|1-\gamma\exp(-2i\pi s)\right|^2\right)^{-1} \\ &\quad \times \left|1-\gamma\exp(-2i\pi s)\right|^{-2} \mathrm{d}s \end{aligned} \tag{6.10}$$

The limit as λ goes to zero yields

$$\ell = \frac{(1-\gamma^2)^2}{4}\int_0^1 \left|1-\gamma\exp(-2i\pi s)\right|^{-4} \mathrm{d}s \tag{6.11}$$

Standard complex analysis gives us the value of ℓ. Indeed, let us denote by $\mathcal{C}$ the circle $\{z \in \mathbb{C};\ |z| = 1\}$. ℓ is equal to

$$\frac{(1-\gamma^2)^2}{8\gamma^2 i\pi}\int_{\mathcal{C}} \frac{z\,\mathrm{d}z}{(z-\gamma)^2(z-\gamma^{-1})^2}$$

and

$$\frac{z}{(z-\gamma)^2(z-\gamma^{-1})^2}=\frac{a}{z-\gamma}+\frac{b}{(z-\gamma)^2}+\frac{c}{z-\gamma^{-1}}+\frac{d}{(z-\gamma^{-1})^2} \quad (6.12)$$

where a, b, c, d are complex constants. It is easy to see that $a=\dfrac{\gamma^2(\gamma^2+1)}{(1-\gamma^2)^3}$ and that the three last terms of the right-hand side of (6.12) have a null integral over $\mathcal{C}$. Hence, by Cauchy formula, we have

$$\ell=\lim_{\lambda\to 0}\langle\eta(0),(\lambda-\mathcal{L}_\beta)^{-1}\eta(0)\rangle=\frac{1}{4}\left(\frac{1+\gamma^2}{1-\gamma^2}\right). \quad \blacksquare \quad (6.13)$$

Let us now consider the general case. We will say that two functions depending on some parameters and β are equivalent and we will note $f\asymp g$ if there exist some positive constants $c, C>0$, independent of β, such that $cf(\beta)\leqslant g(\beta)\leqslant Cf(\beta)$.

In the general case, the assumptions on the flip rates show that :

$$\sum_{x\in\mathbb{Z}^d}\left\langle\left[f(\eta^x)-f(\eta)\right]^2\right\rangle\asymp\mathcal{D}_\beta(f)$$

Consequently, the Laplace transform $\Gamma_\beta(\lambda)$ of the time t variance is equivalent to $(1+\gamma^2)/(1-\gamma^2)\asymp\exp(2\beta)$. But $\Gamma_\beta(\lambda)$ is a decreasing function of λ and using a classical tauberian (Theorem 2, XIII.5 of ref. 3), we have the following corollary.

Corollary 6.3. If the flip rates $\{c_\beta(x,\cdot)|x\in\mathbb{Z}\}$ satisfy the detailed balance condition (6.2), then the following limit exists

$$\sigma_\infty^2=\lim_{t\to+\infty}t^{-1}\sigma_t^2$$

and

$$\sigma_\infty^2\asymp\exp(2\beta)$$

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